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Symmetry of charged rotating body metrics

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Abstract. The maximal local symmetry associated with the exterior metric of a charged rotating object, relating distinct solutions of the field equations, is proved to be eight-parametric. The finite transformations are explicitly exhibited.

1. Introduction

The exterior metrics of charged rotating objects and charged black holes are described by the pseudo-stationary axially symmetric solutions of the Einstein–Maxwell equations, satisfying appropriate asymptotic conditions (Carter 1972). The latter have been cast (Ernst 1968) into a convenient form, in terms of two 'potentials' ϵ and ψ, which are complex functions of two coordinates (ρ, z). The potentials satisfy a set of non-linear partial differential equations:

$$f\nabla^2\epsilon - \nabla\epsilon . (\nabla\epsilon + 2\psi^*\nabla\psi) = 0 \qquad f\nabla^2\psi - \nabla\psi . (\nabla\epsilon + 2\psi^*\nabla\psi) = 0, \tag{1}$$

where

$$2f = \epsilon + \epsilon^* + 2\psi\psi^*, \qquad \text{and} \qquad \nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}.$$

Due to their astrophysical relevance, these equations have been intensively studied, and a few physically accepted solutions were found (Newman *et al* 1965, Esposito and Witten 1973, Tanabe 1977) by various methods. One of the successful approaches is the generation of new solutions out of known solutions (Kinnersley 1977). This can be done with the aid of the local symmetry group associated with the equations, a group which we are going to investigate.

2. The group of transformations

Thus we focus our attention on the most general one-parameter group of transformations

$$\epsilon \to \epsilon'(t) \qquad \psi \to \psi'(t), \tag{2}$$

which transform solutions into solutions. In order to obtain it, the trajectories of (2) are considered in their differential form:

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t}=E(\rho, z, \epsilon, \psi, \epsilon^*, \psi^*) \qquad \frac{\mathrm{d}\psi}{\mathrm{d}t}=\Psi(\rho, z, \epsilon, \psi, \epsilon^*, \psi^*). \tag{3}$$

We seek the most general functions E and Ψ allowed thereby. First, we calculate the dependence on the parameter of the quantities $\nabla\epsilon$, $\nabla\psi$, $\nabla^2\epsilon$ etc, induced by (3):

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\epsilon}{\partial\rho}=\frac{\partial E}{\partial\rho}+\frac{\partial E}{\partial\epsilon}\frac{\partial\epsilon}{\partial\rho}+\frac{\partial E}{\partial\psi}\frac{\partial\psi}{\partial\rho}+\frac{\partial E}{\partial\epsilon^*}\frac{\partial\epsilon^*}{\partial\rho}+\frac{\partial E}{\partial\psi^*}\frac{\partial\psi^*}{\partial\rho}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\psi}{\partial\rho}=\frac{\partial\Psi}{\partial\rho}+\frac{\partial\Psi}{\partial\epsilon}\frac{\partial\epsilon}{\partial\rho}+\frac{\partial\Psi}{\partial\psi}\frac{\partial\psi}{\partial\rho}+\frac{\partial\Psi}{\partial\epsilon^*}\frac{\partial\epsilon^*}{\partial\rho}+\frac{\partial\Psi}{\partial\psi^*}\frac{\partial\psi^*}{\partial\rho}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial^2\epsilon}{\partial\rho^2}=&\frac{\partial^2E}{\partial\rho^2}+2\frac{\partial^2E}{\partial\rho\,\partial\epsilon}\frac{\partial\epsilon}{\partial\rho}+2\frac{\partial^2E}{\partial\rho\,\partial\psi}\frac{\partial\psi}{\partial\rho}+2\frac{\partial^2E}{\partial\rho\,\partial\epsilon^*}\frac{\partial\epsilon^*}{\partial\rho}+2\frac{\partial^2E}{\partial\rho\,\partial\psi^*}\frac{\partial\psi^*}{\partial\rho}\\
&+\frac{\partial^2E}{\partial\epsilon^2}\left(\frac{\partial\epsilon}{\partial\rho}\right)^2+2\frac{\partial^2E}{\partial\epsilon\,\partial\psi}\frac{\partial\epsilon}{\partial\rho}\frac{\partial\psi}{\partial\rho}+\frac{\partial^2E}{\partial\psi^2}\left(\frac{\partial\psi}{\partial\rho}\right)^2+2\frac{\partial^2E}{\partial\epsilon\,\partial\epsilon^*}\frac{\partial\epsilon}{\partial\rho}\frac{\partial\epsilon^*}{\partial\rho}\\
&+2\frac{\partial^2E}{\partial\epsilon\,\partial\psi^*}\frac{\partial\epsilon}{\partial\rho}\frac{\partial\psi^*}{\partial\rho}+2\frac{\partial^2E}{\partial\psi\,\partial\epsilon^*}\frac{\partial\psi}{\partial\rho}\frac{\partial\epsilon^*}{\partial\rho}+2\frac{\partial^2E}{\partial\psi\,\partial\psi^*}\frac{\partial\psi}{\partial\rho}\frac{\partial\psi^*}{\partial\rho}+\frac{\partial^2E}{\partial\epsilon^{*2}}\left(\frac{\partial\epsilon^*}{\partial\rho}\right)^2\\
&+2\frac{\partial^2E}{\partial\epsilon^*\,\partial\psi^*}\frac{\partial\epsilon^*}{\partial\rho}\frac{\partial\psi^*}{\partial\rho}+\frac{\partial^2E}{\partial\psi^{*2}}\left(\frac{\partial\psi^*}{\partial\rho}\right)^2+\frac{\partial E}{\partial\epsilon}\frac{\partial^2\epsilon}{\partial\rho^2}+\frac{\partial E}{\partial\psi}\frac{\partial^2\psi}{\partial\rho^2}\\
&+\frac{\partial E}{\partial\epsilon^*}\frac{\partial^2\epsilon^*}{\partial\rho^2}+\frac{\partial E}{\partial\psi^*}\frac{\partial^2\psi^*}{\partial\rho^2}.\end{aligned}$$

Similar expressions are obtained for the remaining derivatives. Equations (1) are now differentiated with respect to the parameter t, and the aformentioned expressions are substituted. In the resulting expansions, the differential coefficients $\partial\epsilon/\partial\rho$, $\partial^2\epsilon^*/\partial\rho^2$ etc, are subject only to the constraints of equations (1) and their complex conjugates. Thus we can use these equations to eliminate some of the coefficients, ending up with expansions with independent coefficients. Equating to zero the relevant terms, after some rearrangement we are led to the following set of necessary and sufficient conditions to be satisfied by E and Ψ:

$$\nabla^2E=0$$

$$\frac{\partial}{\partial\rho}\left(f\frac{\partial E}{\partial\epsilon}-E-\psi^*\Psi\right)=0, \qquad \frac{\partial}{\partial z}\left(f\frac{\partial E}{\partial\epsilon}-E-\psi^*\Psi\right)=0$$

$$\frac{\partial}{\partial\rho}\left(f\frac{\partial E}{\partial\psi}-\psi^*E\right)=0, \qquad \frac{\partial}{\partial z}\left(f\frac{\partial E}{\partial\psi}-\psi^*E\right)=0$$

$$f^2\frac{\partial^2E}{\partial\epsilon^2}-f\left(\frac{\partial E}{\partial\epsilon}+2\psi^*\frac{\partial\Psi}{\partial\epsilon}\right)+R=0$$

$$\frac{\partial^2E}{\partial\psi^2}=0$$

$$f^2\frac{\partial^2E}{\partial\epsilon\,\partial\psi}-f\left(\frac{1}{2}\frac{\partial E}{\partial\psi}+\psi^*\frac{\partial\Psi}{\partial\psi}+\Psi^*\right)+\psi^*R=0$$

$$\nabla^2\Psi = 0$$

$$\frac{\partial}{\partial\rho}\left(2f\frac{\partial\Psi}{\partial\epsilon}-\Psi\right)=0, \qquad \frac{\partial}{\partial z}\left(2f\frac{\partial\Psi}{\partial\epsilon}-\Psi\right)=0$$

$$\frac{\partial}{\partial\rho}\left(f\frac{\partial\Psi}{\partial\psi}-\tfrac{1}{2}E-2\psi^*\Psi\right)=0, \qquad \frac{\partial}{\partial z}\left(f\frac{\partial\Psi}{\partial\psi}-\tfrac{1}{2}E-2\psi^*\Psi\right)=0$$

$$\frac{\partial^2\Psi}{\partial\epsilon^2}=0$$

$$f^2\frac{\partial^2\Psi}{\partial\psi^2}-f\left(\frac{\partial E}{\partial\psi}+2\psi^*\frac{\partial\Psi}{\partial\psi}+2\Psi^*\right)+2\psi^*R=0$$

$$2f^2\frac{\partial^2\Psi}{\partial\epsilon\,\partial\psi}-f\left(\frac{\partial E}{\partial\epsilon}+2\psi^*\frac{\partial\Psi}{\partial\epsilon}\right)+R=0$$

with

$$R=\tfrac{1}{2}(E+E^*)+\psi^*\Psi+\psi\Psi^*.$$

Furthermore, it can be shown that the local symmetry requirement (with the aid of the equations above) implies that E and Ψ are independent of ϵ^* and ψ^*.

3. Solutions

A lengthy manipulation of the conditions obtained in the last section leads to the most general solution:

$$\begin{aligned}E&=\mathrm{i}A\epsilon^2+2\alpha\epsilon\psi+(\beta+\beta^*)\epsilon-2\gamma\psi+\mathrm{i}B\\ \Psi&=\mathrm{i}A\epsilon\psi+2\alpha\psi^2+\alpha^*\epsilon+\beta\psi+\gamma^*,\end{aligned}\tag{4}$$

with arbitrary real constants A, B; and arbitrary complex constants α, β, γ. Consequently, the maximal local symmetry group is eight-parametric.

In view of (4), equation (3) can be integrated to yield the finite local transformations. It proves more convenient to express the result in terms of the quantities

$$\xi=\frac{1+\epsilon}{1-\epsilon}, \qquad \eta=\frac{2\psi}{1-\epsilon},$$

with the inverse relation

$$\epsilon=\frac{\xi-1}{\xi+1}, \qquad \psi=\frac{\eta}{\xi+1}.$$

These new potentials satisfy a set of equations wherein, unlike (1), the two potentials appear in a symmetrical manner, viz.:

$$(\xi\xi^*+\eta\eta^*-1)\nabla^2\xi-2\boldsymbol{\nabla}\xi.(\xi^*\boldsymbol{\nabla}\xi+\eta^*\boldsymbol{\nabla}\eta)=0$$

$$(\xi\xi^*+\eta\eta^*-1)\nabla^2\eta-2\boldsymbol{\nabla}\eta.(\xi^*\boldsymbol{\nabla}\xi+\eta^*\boldsymbol{\nabla}\eta)=0.$$

The maximal continuous local symmetry group is then:

$$\xi' = \frac{a_{11}\xi + a_{12}\eta + a_{13}}{a_{31}\xi + a_{32}\eta + a_{33}} \qquad \eta' = \frac{a_{21}\xi + a_{22}\eta + a_{23}}{a_{31}\xi + a_{32}\eta + a_{33}}, \tag{5}$$

where

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is an arbitrary SU(2, 1) matrix (the role played here by the group SU(2, 1) has already been recognised (Kinnersley 1973)).

Special cases of formula (5) have been derived (Meinhardt and Leibowitz 1978) (in ($\epsilon - \psi$)-language) as tools for generating new physical solutions starting with physical metrics.

References

Carter B 1972 *Black Holes* eds C DeWitt and B S DeWitt (New York: Gordon and Breach).
Ernst F J 1968 *Phys. Rev.* **168** 1415
Esposito F P and Witten L 1973 *Phys. Rev.* D **8** 3302.
Kinnersley W 1973 *J. Math. Phys.* **14** 651
—— 1977 *J. Math. Phys.* **18** 1529, 1538
Meinhardt J and Leibowitz E 1978 *Phys. Lett.* A to be published
Newman E T, Couch E, Chinnapared K, Exton A, Prakash A and Torrence R 1965 *J. Math Phys.* **6** 918
Tanabe Y 1977 *Prog. Theor. Phys.* **57** 840